Refinement of the paper gCanonical formalism of the f(R)-type gravity in terms of Lie derivativesh

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Abstract

We refine the presentation of the previous paper of our group, Y.Ezawa et al., Class. and Quantum Grav. **23** (2006), 3205. In that paper, we proposed a canonical formalism of f(R)-type generalized gravity by using the Lie derivatives instead of the time derivatives. However, the use of the Lie derivatives was not sufficient. In this note, we make use of the Lie derivatives as far as possible, so that no time derivatives are used.

Introduction

Since the use of the f(R)-type gravity by Caroll et al.[1] to explain the discovered accelerated expansion of the universe[2], the theory has been attracting much attention and various aspects and applications have been investigated[3]. However, its canonical formalism had not been so systematic and useful. So in [4], our group proposed a formalism by generalizing the canonical formalism of Ostrogradski[5]. The generalization is necessary. Since the scalar curvature R depends on the time derivatives of the lapse function and shift vector, these variables have to obey the field equations, if the Ostrogradski's method is directly applied. Then, only the solutions of these equations are allowed for these variables. This, however, is in conflict with general covariance since these variables specify the coordinate frame so should be taken arbitrarily.

In this note, we refine the presentation of our previous paper[4]. In that paper, we proposed a canonical formalism of the f(R)-type gravity in terms of the Lie derivatives by naturally and economically generalizing the formalism of Ostrogradski. However the use of the Lie derivatives was not sufficient, i.e., Lie derivatives and time derivatives were used in a mixed way. Here, we make use of the Lie derivatives as far as possible.

Action

We start from the following action of the generalized gravity of f(R)-type;

$$S = S_G + S_M = \int d^4x\sqrt{-g}f(R) + S_M, \tag{1}$$

where S_M is the action of matters. In other words, the Lagrangian density for gravity $\mathcal{L}_G$ is expressed as

$$\mathcal{L}_G = \sqrt{-g}f(R). \tag{2}$$

As the variables for gravity, we adopt the ADM variables[6]. Then the scalar curvature, R,

is expressed as follows:

$$R = h^{ij}\mathcal{L}_n^{\,2}h_{ij} + \frac{1}{4}\left(h^{ij}\mathcal{L}_n h_{ij}\right)^2 - \frac{3}{4}h^{ik}h_{jl}\mathcal{L}_n h_{ij}\mathcal{L}_n h_{kl} + {}^3R - 2N^{-1}\Delta N, \tag{3}$$

where h_{ij} is the metric of the hypersurface Σ_t which has the normal vector field $n^\mu = N^{-1}(1, -N^i)$, N is the lapse function and N^i is the shift vector. $\mathcal{L}_n$ represents the Lie derivative along the normal vector field n. ${}^3R$ is the scalar curvature of Σ_t. From (2) and (3), $\mathcal{L}_G$ depends on the ADM variables in the following way.

$$\mathcal{L}_G = \mathcal{L}_G(N, h_{ij}, \mathcal{L}_n h_{ij}, \mathcal{L}_n^{\,2}h_{ij}). \tag{4}$$

Variation

From the expression (4), variation of $\mathcal{L}_G$ is expressed as

$$\delta\mathcal{L}_G = \frac{\delta\mathcal{L}_G}{\delta N}\delta N + \frac{\delta\mathcal{L}_G}{\delta h_{ij}}\delta h_{ij} + \frac{\partial\mathcal{L}_G}{\partial(\mathcal{L}_n h_{ij})}\delta\mathcal{L}_n h_{ij} + \frac{\partial\mathcal{L}_G}{\partial(\mathcal{L}_n^{\,2}h_{ij})}\delta\mathcal{L}_n^{\,2}h_{ij}. \tag{5}$$

Here of the first two terms on the right-hand are not the partial derivatives but the functional derivatives since the scalar curvature R depends on the derivatives of N and h_{ij} in ΔN and ${}^3R$ as seen in (3). Actual calculation is made easier when concrete form (3) is used. Then we have

$$\delta\mathcal{L}_G = \delta\sqrt{h}Nf(R) + \sqrt{h}\delta Nf(R) + \sqrt{h}Nf'(R)\delta R, \tag{6}$$

where

$$\left\{\begin{array}{lcl} \delta\sqrt{h} & = & \frac{1}{2}\sqrt{h}h^{ij}\delta h_{ij}, \\ \delta R & = & h^{ij}\mathcal{L}_n^2\delta h_{ij} + (h^{ij}K - 3K^{ij})\mathcal{L}_n\delta h_{ij} + \left[-h^{ik}h^{jl}(\mathcal{L}_n + K)(\mathcal{L}_n k_{kl}) + 6K^{il}K^j{}_l\right]\delta h_{ij} \\ & & +\delta\,{}^3R + 2N^{-2}\Delta N\delta N - 2N^{-1}\Delta\delta N. \end{array}\right. \tag{7}$$

Note that $\delta\mathcal{L}_n h_{ij} = \mathcal{L}_n\delta h_{ij},\ \delta\mathcal{L}_n^{\,2}h_{ij} = \mathcal{L}_n^{\,2}\delta h_{ij}$ and also

$$\left\{\begin{array}{lcl} \sqrt{h}Nf'(R)\,\delta\,{}^3R & = & -\sqrt{h}Nf'(R)R^{ij}\delta h_{ij} \\ & & +\sqrt{h}\left[(Nf'(R))^{;i}\,h^{kl}\Gamma^j_{kl} - (Nf'(R))^{;l}\,h^{ik}\Gamma^j_{kl}\right]\delta h_{ij} \\ & & +\partial_k\left[\sqrt{h}\left\{h^{ik}\,(Nf'(R))^{;j} - h^{ij}\,(Nf'(R))^{;k}\right\}\right]\delta h_{ij} \\ & & +\partial_i\left[\sqrt{h}Nf'(R)\left(h^{kl}\,\delta\Gamma^i_{kl} - h^{il}\,\delta\Gamma^k_{lk}\right)\right] \\ & & -\partial_i\left[\sqrt{h}\left\{h^{ij}\,(Nf'(R))^{;k}\,\delta h_{kj} - (Nf'(R))^{;i}\,h^{kl}\delta h_{kl}\right\}\right], \\ \sqrt{h}f'(R)\Delta\delta N & = & \sqrt{h}\Delta f'(R)\delta N + \partial_k\left[\sqrt{h}(f'(R)\nabla^k\delta N - \nabla^k f'(R)\,\delta N)\right]. \end{array}\right. \tag{8}$$

When we use (7) in (6) and apply the variational principle, gpartial integrationsh have to be done for terms including $\mathcal{L}_n\delta h_{ij}$ and $\mathcal{L}_n^{\,2}\delta h_{ij}$. This is done by using a relation for a scalar field Φ:

$$\mathcal{L}_n(\sqrt{h}N\Phi) = \mathcal{L}_n(\sqrt{h}N)\,\Phi + \sqrt{h}\,N\mathcal{L}_n\Phi, \quad \mathcal{L}_n\Phi = n^\mu\partial_\mu\Phi. \tag{9}$$

Then we have

$$\sqrt{h}Nf'(R)(h^{ij}K-3K^{ij})\mathcal{L}_n\delta h_{ij} \;= -\sqrt{h}N(\mathcal{L}_n+K)\left[f'(R)(h^{ij}K-3K^{ij})\right]\delta h_{ij} + \partial_\mu\left[n^\mu\sqrt{h}Nf'(R)(h^{ij}K-3K^{ij})\delta h_{ij}\right], \tag{10}$$

and

$$\sqrt{h}Nf'(R)h^{ij}\mathcal{L}_n^2\delta h_{ij} \;= \sqrt{h}N\left[K^2f'(R)h^{ij}+\mathcal{L}_n^2(f'(r)h^{ij})+K\mathcal{L}_n(f'(R)h^{ij})\right]\delta h_{ij} + \partial_\mu\left[n^\mu\sqrt{h}N\{f'(R)h^{ij}\delta\mathcal{L}_nh_{ij}-(\mathcal{L}_n+K)(f'(R)h^{ij})\delta h_{ij}\}\right]. \tag{11}$$

Using these relations, we have for $\delta\mathcal{L}_G$ the following expression:

$$\begin{aligned}\delta\mathcal{L}_G \;&= \sqrt{h}\left[\Delta f'(R)+2N^{-1}\Delta Nf'(R)\right]\delta N+\partial_i\left[\sqrt{h}\left(f'(R)\nabla^i\delta N-\nabla^if'(R)\delta N\right)\right]\\ &+\Big[f'''(R)(\mathcal{L}_nR)^2h^{ij}+f''(R)\left(\mathcal{L}_n^2R\,h^{ij}-\mathcal{L}_nK^{ij}\right)\\ &+f'(R)\Big(KK^{ij}-h^{ij}\mathcal{L}_nK-h^{ik}h^{jl}\mathcal{L}_nK_{kl}+6K^{ik}K^j{}_k+\frac{\delta\,{}^3R}{\delta h_{ij}}\Big)+\frac{1}{2}f(R)h^{ij}\Big]\delta h_{ij}\\ &+\partial_\mu\left[n^\mu\sqrt{h}N\left\{f'(R)h^{ij}\delta\mathcal{L}_nh_{ij}-\left(f''(R)\mathcal{L}_nRh^{ij}+f'(R)K^{ij}\right)\delta h_{ij}\right\}\right].\end{aligned} \tag{12}$$

New generalized coordinates and momenta canonically conjugate to them

New generalized coordinates, denoted as Q_{ij}, are taken to be

$$Q_{ij}\equiv\frac{1}{2}\mathcal{L}_nh_{ij}=K_{ij}. \tag{13}$$

Momenta canonically conjugate to the original and new generalized coordinates, p^{ij} and P^{ij} respectively, are defined to be the coefficient of their variations in the time derivative in (12):

$$\begin{cases} p^{ij}=-\sqrt{h}\left[\mathcal{L}_nf'(R)h^{ij}+f'(R)Q^{ij}\right], \\ P^{ij}=2\sqrt{h}f'(R)h^{ij}, \end{cases} \tag{14}$$

where, of course, $\mathcal{L}_nf'(R)$ is also expressed as $f''(R)\mathcal{L}_nR$. Expressions of these equations that correspond to (5) read as follows [1]:

$$p^{ij}=n^0\frac{\partial\mathcal{L}_G}{\partial(\mathcal{L}_nh_{ij})}-\mathcal{L}_n\Big(n^0\frac{\partial\mathcal{L}_G}{\partial(\mathcal{L}_n^2h_{ij})}\Big),\quad P^{ij}=2n^0\frac{\partial\mathcal{L}_G}{\partial(\mathcal{L}_n^2h_{ij})}. \tag{15a}$$

Or reversingly, we have

$$\frac{\partial\mathcal{L}_G}{\partial(\mathcal{L}_nh_{ij})}=\frac{1}{n^0}\Big(p^{ij}+\frac{1}{2}\mathcal{L}_nP^{ij}\Big),\quad \frac{\partial\mathcal{L}_G}{\partial(\mathcal{L}_n^2h_{ij})}=\frac{1}{2n^0}P^{ij}. \tag{15b}$$

[1] If we use a scalar function $\tilde{\mathcal{L}}_G$ defined as $\mathcal{L}_G\equiv N\sqrt{h}\tilde{\mathcal{L}}_G$, factors n^0 disappear and we have

$$p^{ij}=\sqrt{h}\frac{\partial\tilde{\mathcal{L}}_G}{\partial(\mathcal{L}_nh_{ij})}-\mathcal{L}_n\Big(\sqrt{h}\frac{\partial\tilde{\mathcal{L}}_G}{\partial(\mathcal{L}_n^2h_{ij})}\Big),\quad P^{ij}=\sqrt{h}\frac{\partial\tilde{\mathcal{L}}_G}{\partial(\mathcal{L}_n^2h_{ij})}.$$

Hamiltonian density

Correspondence of each point on different Σ_t are given by a 1-parameter transformation along the timelike vector field t^μ, we have, e.g.,

$$h_{ij}(\mathbf{x}, t + \delta t) = h_{ij}(\mathbf{x}, t) + \mathcal{L}_t h_{ij} \delta t. \tag{16}$$

Actually, we have $\partial_0 h_{ij} = \mathcal{L}_t h_{ij}$ in the coordinate frame we are using. Thus Hamiltonian density $\mathcal{H}_G$ is defined to be

$$\mathcal{H}_G \equiv p^{ij} \mathcal{L}_t h_{ij} + P^{ij} \mathcal{L}_t Q_{ij} - \mathcal{L}_G. \tag{17}$$

Invariance of the Hamiltonian

We consider the following transformations of the generalized coordinates h_{ij}:

$$h_{ij} \to \phi_{ij} \equiv F_{ij}(h_{kl}) \quad \text{or inversely} \quad h_{ij} \equiv G_{ij}(\phi_{kl}), \tag{18}$$

and show that the Hamiltonian is invariant under this transformation. New generalized coordinates Φ_{ij} are defined as in (13), i.e.,

$$\Phi_{ij} \equiv \frac{1}{2} \mathcal{L}_n \phi_{ij}. \tag{19}$$

Hamiltonian density $\bar{\mathcal{H}}_G$ expressed in the transformed variables is defined to be

$$\bar{\mathcal{H}}_G \equiv \pi^{ij} \mathcal{L}_t \phi_{ij} + \Pi^{ij} \mathcal{L}_t \Phi_{ij} - \bar{\mathcal{L}}_G(N, \phi_{ij}, \mathcal{L}_n \phi_{ij}, \mathcal{L}_n^2 \phi_{ij}), \tag{20}$$

where π^{ij} and Π^{ij} are momenta canonically conjugate to ϕ_{ij} and Φ_{ij}, respectively, and since

$$\mathcal{L}_n h_{ij} = \frac{\partial G_{ij}}{\partial \phi_{kl}} \mathcal{L}_n \phi_{kl}, \quad \mathcal{L}_n^2 h_{ij} = \mathcal{L}_n \Big(\frac{\partial G_{ij}}{\partial \phi_{kl}} \Big) \mathcal{L}_n \phi_{kl} + \frac{\partial G_{ij}}{\partial \phi_{kl}} \mathcal{L}_n^2 \phi_{kl}, \tag{21}$$

$\bar{\mathcal{L}}_G$ is defined as

$$\bar{\mathcal{L}}_G(N, \phi_{ij}, \mathcal{L}_n \phi_{ij}, \mathcal{L}_n^2 h_{ij}) \equiv \mathcal{L}_G \left(N, G_{ij}(\phi_{kl}), \frac{\partial G_{ij}}{\partial \phi_{kl}} \mathcal{L}_n \phi_{kl}, \mathcal{L}_n \Big(\frac{\partial G_{ij}}{\partial \phi_{kl}} \Big) \mathcal{L}_n \phi_{kl} + \frac{\partial G_{ij}}{\partial \phi_{kl}} \mathcal{L}_n^2 \phi_{kl} \right). \tag{22}$$

π^{ij} and Π^{ij} satisfy relations similar to (15a,b), and from these relations, we have

$$\pi^{ij} = p^{kl} \frac{\partial G_{kl}}{\partial \phi_{ij}}, \quad \Pi^{ij} = P^{kl} \frac{\partial G_{kl}}{\partial \phi_{ij}}, \tag{23a}$$

or inversely

$$p^{ij} = \pi^{kl} \frac{\partial F_{kl}}{\partial h_{ij}}, \quad P^{ij} = \Pi^{kl} \frac{\partial F_{kl}}{\partial h_{ij}}. \tag{23b}$$

With help of (23a,b), we have

$$p^{ij} \mathcal{L}_t h_{ij} = \pi^{kl} \frac{\partial F_{kl}}{\partial h_{ij}} \frac{\partial F_{ij}}{\partial \phi_{mn}} \mathcal{L}_t \phi_{mn} = \pi^{ij} \mathcal{L}_t \phi_{ij}. \tag{24}$$

Similar relation holds between P^{ij} and Π^{ij}, so we have

$$\mathcal{H}_G = \bar{\mathcal{H}}_G. \tag{25}$$

It is noted that the transformation (18) includes the coordinate transformation on Σ_t.

Summary

We presented a canonical formalism of $f(R)$-type gravity in terms of the Lie derivatives by refining our previous paper[4]. The formalism is a natural and economical generalization of the Ostrogradski's formalism. Generalization is necessary to assure the invariance of the theory under the general coordinate transformation.

References

[1] S. Caroll, V. Duvuri, M. Trodden and M. S. Turner, *Phys. Rev.* **D70** (2004), 043528

[2] B. N. Reid et al., *MNRAS* **404** (2010),60
W. J. Percival et al., *MNRAS* **401** (2009), 2148
M. Hicken et al., *ApJ* **700** (2009), 1097
R. Kessler et al., *ApJS***185** (2009), 32
V. Vikhlini et al., *ApJ* **692** (2009), 1033
A. Mantz et al., *MNRAS* **406** (2010), 1759
A. G. Riess et al., *ApJ* **699** (2009), 539
S. H. Suyu et al., arXiv: 0910.2773 [astro-ph.CO]
R. Fadely et al., *ApJ* **711** (2009), 211
R. Masset et al., *ApJS* **172** (2007), 239
L. Fu et al., *Astron. Astrophys.* **479** (2009), 9
T. Schrabback et al., *Astron. Astrophys.* **516** (2009), 63

[3] See for example, T. P. Sotiriou and V. Faraoni, *Rev. Mod. Phys.* **82** (2008), 451
S. Nojiri and S.Odintsov, *Int.J.Geom.Meth.Phys.* **4** (2007), 115

[4] Y. Ezawa, H. Iwasaki, Y. Ohkuwa, S. Watanabe, N. Yamada and T. Yano, *Class. Quantum Grav.* **23** (2006), 3205

[5] M. Ostrogradski, *Mem. Acad. Sci. St. Petersberg* VI **4** (1850), 385

[6] R. Arnowitt, S. Deser and C. Misner, arXiv:gr-qc/0405109 gThe Dynamics of General Relativityh(2004)